

Three-Dimensional Chern-Simons and BF Theories

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Abstract

Our aim in this note is to clarify a relationship between covariant Chern-Simons 3-dimensional theory and Schwartz type topological field theory known also as BF theory.

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1 Introduction.

Chern-Simons theory gives an interesting example of topological field theory. Its Lagrangian 3-form lives on a principal G -bundle and after pulling back to space-time (base) manifold provides, in general, a family of local, non-covariant Lagrangian densities [3, 4].¹ Because of this, it is also more difficult to analyze, in this case, Nöther conserved quantities [2]. A more standard approach to the problem of symmetries and conservation laws has been applied in the so called covariant formalism [3]. It exploits the transgression 3-form as a global and covariant Chern-Simons Lagrangian with two dynamical gauge fields. This formalism has been used for the calculation of Nöther currents and their identically vanishing parts - superpotentials. Augmented variational principle and relative conservation laws have been recently proposed in [6]. Our aim in the present note, which can be viewed as an appendix to [3], is to explain a link between covariant Chern-Simons theory and the so called BF theories [1, 5, 7].

¹However, the corresponding Euler-Lagrange equations of motion have well-defined global meaning.

2 Change of variables.

Let us consider a principal bundle $P(M, G)$ over a three-dimensional manifold M with a (semisimple) structure group G . Let ω_i ($i = 0, 1$) be two principal connection 1-forms with the corresponding curvature 2-forms

$$\Omega_i = d\omega_i + \omega_i^2 = d\omega_i + \frac{1}{2}[\omega_i, \omega_i]. \quad (1)$$

Denote by $\alpha = \omega_1 - \omega_0$, a tensorial 1-form.

The transgression 3-form is given by the well known formula

$$\begin{aligned} Q(\omega_1, \omega_0) &= -Q(\omega_0, \omega_1) = \text{tr} \left(2\Omega_0 \wedge \alpha + D_0\alpha \wedge \alpha + \frac{2}{3}\alpha^3 \right) \\ &= \text{tr} \left(2\Omega_1 \wedge \alpha - D_1\alpha \wedge \alpha + \frac{2}{3}\alpha^3 \right) \end{aligned} \quad (2)$$

where $D_i\alpha = d\alpha + [\omega_i, \alpha]$ denotes the covariant derivative of α with respect to the connection ω_i . Thus $Q(\omega_1, \omega_0)$ is a tensorial (covariant) object which well-defines the corresponding global 3-form on M . It undergoes a non-covariant splitting as a difference of two Chern-Simons Lagrangians

$$Q(\omega_1, \omega_0) = CS(\omega_1) - CS(\omega_0) + d\text{tr}(\omega_0 \wedge \omega_1) \quad (3)$$

where

$$CS(\omega) = \text{tr} \left(\Omega \wedge \omega - \frac{1}{3}\omega^3 \right) \quad (4)$$

stands for non-covariant Chern-Simons Lagrangian.

Notice that in the case of two connections one has

$$\begin{aligned} 2\Omega_0 + D_0\alpha &= 2\Omega_1 - D_1\alpha = \\ \Omega_0 + \Omega_1 + \frac{1}{2}(D_0\alpha - D_1\alpha) &= \Omega_0 + \Omega_1 - \alpha^2 \end{aligned} \quad (5)$$

The last equality entitles us to rewrite

$$Q(\omega_1, \omega_0) = 2\text{tr} \left(\bar{\Omega}\alpha + \frac{1}{12}\alpha^3 \right) \quad (6)$$

where $\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_0)$ is a new (average) connection and $\bar{\Omega} = d\bar{\omega} + \bar{\omega}^2$. Of course, one has $\omega_1 = \bar{\omega} + \frac{1}{2}\alpha$, $\omega_0 = \bar{\omega} - \frac{1}{2}\alpha$.

Thus the Lagrangian $Q(\omega_1, \omega_0)$ can be treated in three different (but equivalent) ways:

- with two (flat) connections ω_0, ω_1 as dynamical variables; see (3). In this case $\Omega_0 = 0$ and $\Omega_1 = 0$ are equations of motion. This point of view was presented in [3].
- with a (flat) connection ω_1 and tensorial 1-form α as dynamical variables; see (2). In this case equations of motion are $\Omega_1 = 0$ and $D_1\alpha = \alpha^2$.
- with an "average" (non-flat) connection $\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_0)$ and tensorial 1-form α as independent dynamical variables; see (6). In this case $\bar{\Omega} = -\frac{1}{4}\alpha^2$ and $\bar{D}\alpha = 0$ are equations of motion. This is the so called BF theory with a cosmological constant $\Lambda = 1$ (see e.g references [1, 5, 8, 9]).

More generally, one can define a new connection $\omega_t = t\omega_1 + (1-t)\omega_0 = \omega_0 + t\alpha$ as a convex combination of two connections with parameter $0 \leq t \leq 1$. The inverse transformation is $\omega_0 = \omega_t - t\alpha$, $\omega_1 = \omega_t + (1-t)\alpha$. In this case

$$\Omega_t = t\Omega_1 + (1-t)\Omega_0 - t(1-t)\alpha^2.$$

Now the equation (5) can be replaced by the more general one

$$\begin{aligned} 2\Omega_1 - D_1\alpha &= 2\Omega_0 + D_0\alpha = \\ &= 2\Omega_t - 2t(1-t)\alpha^2 - (2t-1)D_t\alpha \end{aligned} \quad (7)$$

(notice that $t\omega_1 + (1-t)\omega_0 = (2t-1)\omega_t + 2t(1-t)\alpha$).

In this new variables (ω_t, α) we obtain

$$Q(\omega_1, \omega_0) = 2tr \left(\Omega_t \wedge \alpha - \left(t - \frac{1}{2} \right) D_t\alpha \wedge \alpha + \left(\frac{1}{3} - t + t^2 \right) \alpha^3 \right) \quad (8)$$

The corresponding equations of motion are:

$$\Omega_t = -t(1-t)\alpha^2, \quad D_t\alpha = (2t-1)\alpha^2 \quad (9)$$

Thus the choices $t = \frac{1}{2}, 0, 1$ lead to the simplest formulae.

It is interesting to observe that the superpotential related to (infinitesimal) gauge transformation χ remains independent of the choice of variables (ω_t, α) (compare formula (19) in [3]), i.e.:

$$U(\chi) = tr(\alpha\chi).$$

Instead, an explicit expression for the superpotential related to (infinitesimal) diffeomorphism transformation driven by a vectorfield ξ does depend on the variables (ω_t, α) and equals to (compare formulae (23,25) in [3]):

$$U(\xi) = tr[\alpha(2\omega_t(\xi) + (1-2t)\alpha(\xi))] .$$

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